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The conservation laws and group properties of the equations of gas dynamics with zero velocity of sound☆

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ABSTRACT

All the conservation laws of zero order are obtained by the method of A-operators for a system of n-dimensional ($n \geq 1$) equations of gas dynamics with zero velocity of sound. A group subdivision is carried out of this system with respect to an infinite subgroup, which is a normal divider of its main Lie group of transformations; the main group of the resolving system is obtained. First-order non-local symmetries are obtained for the initial system. A special choice of the mass Lagrange variables enables this system to be converted to a reduced system equivalent to it, containing n - 1 spatial variables, which, for $n = 2$, is written in the form of a one-dimensional complex heat-conduction equation using complex dependent and independent variables.

1. Introduction

A systematic investigation of the submodels of gas dynamics was initiated by Ovsyannikov[1,2] using the “Podmodeli” (“Submodels”) program. One of these submodels, number 13,[2] specified a system of equations of gas dynamics with zero velocity of sound

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{0}, \quad \rho_t + \mathbf{u}\cdot\nabla\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \quad p_t + \mathbf{u}\cdot\nabla p = 0$$

$$\mathbf{x} \in R^n, \quad n \geq 1 \tag{1.1}$$

and will be the main object of our investigation in this paper. Here $\mathbf{u} = \mathbf{u}(t, x) \in R^n$ is the velocity vector, $\rho = \rho(t, \mathbf{x})$ is the density and $p = p(t, \mathbf{x})$ is the pressure.

Below we obtain a complete system of conservation laws of zero order for Eqs (1.1). The presence in the main group of system (1.1) of an infinite normal divider enabled us to carry out a group subdivision[3] of this system. Using a special choice of the mass Lagrange variables we were able to reduce this system, that is, convert it to a system containing a smaller number of independent variables.

2. The A-operator method

Consider an arbitrary system (S) of differential equations for m ($m \geq 1$) required functions $\mathbf{u} = (u^1, u^2, \ldots, u^m)$ of $n+1$ ($n \geq 1$) independent variables $\boldsymbol{y} = (x^0, x^1, x^2, \ldots, x^n)$. Suppose [$S$] is a manifold in extended space, specified by the equations of system (S) and all its differential corollaries.

The vector $\mathbf{A} = \mathbf{A}\left(\mathbf{y}, \mathbf{u}, \underset{1}{\mathbf{u}}, \underset{2}{\mathbf{u}}, \ldots\right) = \left(A^0, A^1, A^2, \ldots, A^n\right)$ such that $(\mathbf{D}\cdot\mathbf{A})_{[s]}=0$, where $\boldsymbol{D} = (D_0, D_1, D_2, \ldots, D_n)$; $D_i = D_x^i$ is the operator of total differentiation with respect to the variable x^i=(i=0, 1, 2, . . ., n), is called[3] the conservation law for system (S).

Definition 1. Suppose **A** is the conservation law of system (S). The evolution operator of generalized symmetry $X = \boldsymbol{\eta}\left(\boldsymbol{y}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}, \underset{2}{\boldsymbol{u}}, \ldots\right)\cdot \partial_{\boldsymbol{u}} + \ldots$, allowed by the equation $(\mathbf{D}\cdot\mathbf{A})=0$ by virtue of system (S) and all its differential corollaries, will be called the **A**-operator of this

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system. Hence, the **A**-operator of the system (S) is specified by the relation

$$(X(\mathbf{D}\cdot\mathbf{A}))_{[S]}=0 \tag{2.1}$$

For the set of **A**-operators of system (S) we can state a lower limit: this set contains a Lie algebra of all the generalized symmetries of system (S).

It was shown in Ref. 4 that the result of the action of each evolution operator of generalized symmetry of system (S) on any of its conservation laws is the conservation law of this system.

The relation between the **A**-operators of the system and its conservation laws is established by the following two propositions.

Proposition 1. *The action of any* **A***-operator of system (S) on the conservation law* ***A*** *gives the conservation law of this system.*

Proposition 2. *Suppose* **A** *is the conservation law of system (S). Any evolution operator of generalized symmetry* $\boldsymbol{X}=\boldsymbol{\eta}\left(\boldsymbol{y},\boldsymbol{u},\underset{1}{\boldsymbol{u}},\underset{2}{\boldsymbol{u}},\ldots\right)\cdot\partial_{\boldsymbol{u}}+\ldots$*, for which the vector X****A*** *is the conservation law of this system, is its* ***A****-operator.*

Definition 2. The **A**-operator X of system (S) will be called its trivial **A**-operator if the vector X**A** is a trivial conservation law of this system.

Definition 3. Two **A**-operators of system (S) will be said to be **A**-equivalent if their difference is a trivial **A**-operator of this system.

The action of **A**-equivalent **A**-operators of system (S) on the conservation law **A** gives equivalent conservation laws of this system. Consequently, the set of **A**-operators of system (S) for each conservation law **A** can be split into classes of **A**-equivalent **A**-operators.

We have the following theorem on the generating conservation law for a system of differential equations.

Theorem 1. *If the system of differential equations (S) has a conservation law of zero order* ***A****, the common rank of the Jacobi matrix* $\partial\boldsymbol{A}/\partial\boldsymbol{u}$ *of which is equal to the number of independent variables of system (S), then each of its conservation laws can be obtained by the action on conservation law* ***A*** *of a certain* **A***-operator of this system.*

The method of obtaining the conservation laws for systems of differential equations using Theorem 1 will be called the **A**-operator method.

3. Conservation laws

In gas dynamics the physical meaning of the conservation law $\boldsymbol{A}=(\boldsymbol{A}^0,\boldsymbol{A}^1,\boldsymbol{A}^2,\ldots,\boldsymbol{A}^n)$ is defined by the component A^0 – the conservation law density, and the flux vector $\mathbf{B}=\mathbf{A}_1-A^0\mathbf{u}$, where $A_1=(A^1,A^2,\ldots,A^n)$.

To find all the conservation laws of zero order for system (1.1) we will take as the generating conservation law **A** the law of conservation of momentum:

$$A^0=\rho\mathbf{u}\cdot\mathbf{b},\quad \mathbf{B}=p\mathbf{b} \tag{3.1}$$

where **b** is fixed unit vector.

Trivial conservation laws of zero order for this system are only functions of the independent variables.

The solution of the system of governing equations (2.1) in this case shows that the set of non-trivial **A**-operators for system (1.1) is generated by the following operators

$$X_1=\frac{1}{\mathbf{u}\cdot\mathbf{b}}\left[\frac{h(p)}{\rho}\mathbf{u}\cdot\partial_{\mathbf{u}}+\left(\rho h'(p)\frac{|\mathbf{u}|^2}{2}-2h(p)\right)\partial_\rho\right]$$

$$X_2=\frac{1}{\mathbf{u}\cdot\mathbf{b}}\left[\frac{1}{\rho}\boldsymbol{\lambda}(p)\cdot\partial_{\mathbf{u}}+\left(\rho\mathbf{u}\cdot\boldsymbol{\lambda}'(p)-\frac{\boldsymbol{\lambda}(p)\cdot\mathbf{b}}{\mathbf{u}\cdot\mathbf{b}}\right)\partial_\rho\right]$$

$$X_3=\frac{1}{\mathbf{u}\cdot\mathbf{b}}\left[\frac{1}{\rho}Q(p)\langle\mathbf{x}\rangle\cdot\partial_{\mathbf{u}}+\left(\rho Q_p(p)\langle\mathbf{x}\rangle\cdot\mathbf{u}-\frac{Q(p)\langle\mathbf{x}\rangle\cdot\mathbf{b}}{\mathbf{u}\cdot\mathbf{b}}\right)\partial_\rho\right]$$

$$X_4=\frac{1}{\mathbf{u}\cdot\mathbf{b}}\left[\frac{t}{\rho}\boldsymbol{\mu}(p)\cdot\partial_{\mathbf{u}}+\left(\rho(t\mathbf{u}-\mathbf{x})\cdot\boldsymbol{\mu}'(p)-\frac{t\boldsymbol{\mu}(p)\cdot\mathbf{b}}{\mathbf{u}\cdot\mathbf{b}}\right)\partial_\rho\right]$$

$$X_5=\frac{\rho g(p)}{\mathbf{u}\cdot\mathbf{b}}\partial_\rho$$

where $h(p)$, $g(p)$, $\lambda(p)$, $\boldsymbol{\mu}(p)$ are arbitrary functions, $Q(p)$ is an arbitrary antisymmetric second-rank tensor in R^n for $n\geq 2$ and $Q(p)=0$ for $n=1$.

The action of the operators X_1, X_2, X_3, X_4, X_5 on conservation law (3.1) gives the following conservation laws for system (1.1) respectively

$$A^0=\rho h'(p)\frac{|\mathbf{u}|^2}{2}-h(p),\quad \mathbf{B}=h(p)\mathbf{u} \tag{3.2}$$

$$A^0=\rho\boldsymbol{\lambda}'(p)\cdot\mathbf{u},\quad \mathbf{B}=\boldsymbol{\lambda}(p) \tag{3.3}$$

$$A^0=\rho Q_p(p)\langle\mathbf{x}\rangle\cdot\mathbf{u},\quad \mathbf{B}=Q(p)\langle\mathbf{x}\rangle \tag{3.4}$$

$$A^0=\rho(t\mathbf{u}-\mathbf{x})\cdot\boldsymbol{\mu}'(p),\quad \mathbf{B}=t\boldsymbol{\mu}(p) \tag{3.5}$$

$$A^0 = \rho g(p), \quad \mathbf{B} = \mathbf{0} \tag{3.6}$$

Here (3.2), (3.3) and (3.4) are generalized energy, momentum and angular momentum conservation laws, (3.5) is a conservation law defining the generalized law of motion of the centre of mass, and (3.6) is the law of conservation of pressure.

When $g(p)\equiv$const conservation law (3.6) is the law of conservation of mass.

Hence, the set of non-trivial conservation laws of zero order for system (1.1) for all $n \geq 1$ consists of conservation laws (3.2)–(3.6).

The group interpretation of conservation laws (3.2)–(3.6) is as follows: the fundamental Lie algebra of the point operators of system (1.1), calculated by the standard method, turns out to be infinitely dimensional and contains an ideal, generated by the operators

$$Y_f = f'(p)\rho\partial_\rho + f(p)\partial p \tag{3.7}$$

where $f = f(p)$ is an arbitrary function. The factoralgebra according to this ideal is finite-dimensional and has the basis

$$\partial_t, \partial_\mathbf{x}, t\partial_\mathbf{x} + \partial_\mathbf{u}, \quad t\partial_t + \mathbf{x}\cdot\partial_\mathbf{x}, \quad t\partial_t - \mathbf{u}\cdot\partial_\mathbf{u} + 2\rho\partial_\rho, \quad \Gamma\langle\mathbf{x}\rangle\cdot\partial_\mathbf{x} + \Gamma\langle\mathbf{u}\rangle\cdot\partial_\mathbf{u} \tag{3.8}$$

(Γ is an arbitrary antisymmetric second-rank tensor in R^n for $n \geq 2$, and $\Gamma = 0$ for $n = 1$).

The action of the operator Y_f on the classical energy, momentum and angular momentum conservation laws, and also on the conservation law defining the law of motion of the centre of mass, known for the equations of gas dynamics,[4,5] gives generalized conservation laws (3.2)–(3.5) respectively. This means that the operator Y_f is an **A**-operator of system (1.1) for all the above-mentioned classical conservation laws.

4. Group subdivision

The presence in the fundamental Lie group of transformations of system (1.1) of an infinite normal divider N, generated by operators (3.7), enables us to carry out an effective group subdivision of this system.

The basis of the first-order differential invariance of group N can be chosen as follows:

$$t,\ \mathbf{x},\ \mathbf{u},\ \frac{1}{\rho^2}(p_t\nabla\rho - \rho_t\nabla p),\ \ \frac{1}{\rho}p_t,\ \ \frac{1}{\rho}\nabla p$$

The automorphous system of group subdivision of system (1.1) with respect to the group N has the form

$$D_t(\rho\mathbf{a}) + D_\mathbf{x}[\rho(\mathbf{a}\cdot\mathbf{u})] = \mathbf{0}, \quad p_t = \rho\mathbf{a}\cdot\mathbf{u}, \quad \nabla p = -\rho\mathbf{a} \tag{4.1}$$

and the resolvent

$$\frac{d\mathbf{u}}{dt} = \mathbf{a}, \quad \mathbf{u}\cdot[\mathbf{a}_t + \nabla(\mathbf{a}\cdot\mathbf{u}) - \mathbf{a}\,\mathrm{div}\,\mathbf{u}] = 0; \quad \frac{d}{dt} = \partial_t + \mathbf{u}\cdot\nabla$$
$$Q\langle\mathbf{a}_t + \nabla(\mathbf{a}\cdot\mathbf{u}) - (\mathbf{a}\cdot\mathbf{u})\nabla\rangle\cdot\mathbf{a} = 0 \tag{4.2}$$

where Q is an arbitrary antisymmetric second-rank tensor in R^n for $n \geq 2$, $Q = 0$ for $n = 1$, and $\mathbf{a} = \mathbf{a}(t, \mathbf{x})$ is the acceleration vector.

Representation of system (1.1) in the form of an amalgamation of systems (4.1) and (4.2), equivalent to it, is also the required group subdivision.

When $n = 1$ the resolvent (4.2) takes the form

$$\frac{du}{dt} = a, \quad \frac{da}{dt} = 0 \tag{4.3}$$

Consequently, for one-dimensional motion, acceleration is conserved in a particle. The introduction of the mass Lagrange variable

$$\xi = \xi(t, x) : \frac{d\xi}{dt} = 0, \xi_x = \rho$$

enables system (4.3) to be integrated, giving

$$a = a_0(\xi), \quad u = ta_0(\xi) + u_0(\xi), \quad x = \frac{t^2}{2}a_0(\xi) + tu_0(\xi) + x_0(\xi), \quad \rho = \frac{1}{x_\xi}$$

where $a_0(\xi)$, $u_0(\xi)$, $x_0(\xi)$ are the initial values of the corresponding quantities. The pressure is found from automorphous system (4.1)

$$p = -\int a_0(\xi)\, d\xi$$

when $n = 2$ and $n = 3$ the resolvent (4.2) is reduced to the form

$$\frac{d\mathbf{u}}{dt} = \mathbf{a}, \quad \mathbf{a}_t + \nabla(\mathbf{a}\cdot\mathbf{u}) - \mathbf{a}\,\mathrm{div}\,\mathbf{u} = \mathbf{u}\Lambda(\nabla\Lambda\mathbf{a}), \quad \mathbf{a}\cdot(\nabla\Lambda\mathbf{a}) = 0 \tag{4.4}$$

When $n = 2$ the last equation of system (4.4) is satisfied identically, and when $n = 3$ it is overdetermined.

The solution of the corresponding system of governing equations shows that the fundamental Lie group of the transformations of the resolvent (4.2) ($n \geq 2$) is finite-dimensional and the basis of its algebra consists of the operators

$$\partial_t, \quad \partial_{\mathbf{x}}, \quad t\partial_{\mathbf{x}} + \partial_{\mathbf{u}}, \quad t\partial_t + \mathbf{x}\cdot\partial_{\mathbf{x}} - \mathbf{a}\cdot\partial_{\mathbf{a}}$$
$$t\partial_t - \mathbf{u}\cdot\partial_{\mathbf{u}} - 2\mathbf{a}\cdot\partial_{\mathbf{a}}, \quad \Gamma\langle\mathbf{x}\rangle\cdot\partial_{\mathbf{x}} + \Gamma\langle\mathbf{u}\rangle\cdot\partial_{\mathbf{u}} + \Gamma\langle\mathbf{a}\rangle\cdot\partial_{\mathbf{a}}$$

(Γ is an arbitrary antisymmetric second-rank tensor in R^n).

Consequently, the fundamental Lie group of transformations of resolvent (4.2) is a continuation on $\mathbf{a} = d\mathbf{u}/dt$ of the generated by operators (3.8) factorgroup of the fundamental group of system (1.1) with respect to the normal divider, generated by operators (3.7).

5. Lagrange variables

Using the mass Lagrange variables $\xi = \xi(t, \mathbf{x})$

$$\frac{d\xi}{dt} = \mathbf{0}, \quad \left|\frac{\partial\xi}{\partial\mathbf{x}}\right| = \rho \tag{5.1}$$

system (1.1) can be written in the form

$$\left|\frac{\partial\mathbf{x}}{\partial\xi}\right|^{-1}\left(\frac{\partial\mathbf{x}}{\partial\xi}\right)^T\mathbf{x}_{tt} + p_{\xi} = \mathbf{0}, \quad p_t = 0 \tag{5.2}$$

The velocity vector and the density are defined by the formulae

$$\mathbf{u} = \mathbf{x}_t, \quad \rho = \left|\frac{\partial\mathbf{x}}{\partial\xi}\right|^{-1} \tag{5.3}$$

We have the following alternative.[2]

1. In system (5.2) we can assume that $p = p(t, \xi)$ is the unknown function. In this case the problem arises of finding the fundamental Lie group of conversions of this system.

2. In system (5.2) the last equation can be integrated: $p=p(\xi)$, where $p(\xi) \neq 0$ is a specified function, defined by the initial pressure distribution. As a result the system takes the form

$$\left|\frac{\partial\mathbf{x}}{\partial\xi}\right|^{-1}\left(\frac{\partial\mathbf{x}}{\partial\xi}\right)^T\mathbf{x}_{tt} + p_{\xi}(\xi) = \mathbf{0} \tag{5.4}$$

In this case the problem arises of reducing system (5.4) to a simpler form, using its equivalence transformations, and investigating the group properties of the system obtained

As already noted, when $n = 1$ system (1.1) is integrated in mass Lagrange variables. We will therefore assume that $n \geq 2$.

The first part of the alternative. The solution of the corresponding system of resolvents shows that the fundamental Lie group of transformations of system (5.2) is generated by the operators

$$\alpha(p)\partial_t, \quad \boldsymbol{\beta}(p)\cdot\partial_{\mathbf{x}}, \quad t\boldsymbol{\gamma}(p)\cdot\partial_{\mathbf{x}}, \quad \Omega(p)\langle\mathbf{x}\rangle\cdot\partial_{\mathbf{x}}, \quad nt\partial_t + 2\boldsymbol{\xi}\cdot\partial_{\xi}$$
$$\sigma(p)[(n-2)t\partial_t - 2\mathbf{x}\cdot\partial_{\mathbf{x}}], \quad \tau(p)\partial_p - \frac{\tau'(p)}{n+2}(2t\partial_t + \mathbf{x}\cdot\partial_{\mathbf{x}}), \quad [\mathrm{div}_{\xi}\Phi(\xi)]\cdot\partial_{\xi} \tag{5.5}$$

where $\alpha(p)$, $\boldsymbol{\beta}(p)$, $\gamma(p)$, $\sigma(p)$, $\tau(p)$ are arbitrary functions and $\Omega(p)$, $\Phi(\xi)$ are arbitrary antisymmetric second-rank tensors in R^n.

The operators (5.5) of the Lagrange system (5.2), continued on $\mathbf{u}$, ρ using formulae (5.3), allow of a system in Euler variables, constructed from Eqs (1.1) and (5.1). The operators obtained are first-order symmetries to the last system, since the coordinates for ∂_p for these operators (with the exception of the trivial operators) depend on the elements of the Jacobi matrix ($\partial\xi/\partial\mathbf{x}$). These operators are non-local first-order symmetries to system (1.1) with a non-local variable ξ.

The second part of the alternative. The equivalence transformations for system (5.4) have the form $\xi'=\mathbf{f}(\xi)$ with any function $\mathbf{f}(\xi)$ satisfying the condition $|\partial\mathbf{f}/\partial\boldsymbol{\xi}|=1$. The arbitrary element $p(\xi)$ is then converted by the formula $p'(\xi')=p(\mathbf{f}^{-1}(\xi'))$.

As a result of the following equivalence transformations

$$\xi'^k = f^k(\xi^1, \xi^2, \ldots, \xi^n), \quad k = 1, 2, \ldots, n-1, \quad \xi'^n = p(\xi^1, \xi^2, \ldots, \xi^n) \tag{5.6}$$

where p is the pressure and $|\partial(f^1, f^1, \ldots, f^{n-1}, p/\partial(\xi^1, \xi^2, \ldots, \xi^n)|=1$, system (5.4) is converted to the following equivalent system (the primes are omitted)

$$\mathbf{x}_{tt} + \left|\frac{\partial\mathbf{x}}{\partial\xi}\right|\left[\left(\frac{\partial\mathbf{x}}{\partial\xi}\right)^T\right]^{-1}\cdot(0, 0, \ldots, 0, 1)^T = \mathbf{0} \tag{5.7}$$

which does not contain the derivative $\partial_{\xi n}\mathbf{X}$.

Hence, the special choice of the mass Lagrange variables enables us to convert system (5.4) to its equivalent reduced system (5.7), containing the (n-1) spatial variables $\xi^1, \xi^2, \ldots, \xi^{n-1}$. We can take as the variable $\xi^n = p$ the parameter on which the solution of system (5.7) depends.

We will henceforth only consider the physically interesting cases when $n=2$ and $n=3$.

When $n=2$ non-linear system (5.4) with the three independent variables t, $\xi^1=\xi$, $\xi^2=\eta$ is reduced to the equivalent linear system of the form (5.7) with two independent variables t and ξ

$$x^1_{tt} = x^2_{\xi}, \quad x^2_{tt} = -x^1_{\xi}; \quad (x^1, x^2) = \mathbf{x} \tag{5.8}$$

The fundamental Lie group of transformations of system (5.8) is infinite, in view of its linearity. It contains a normal divider, generated by the operators

$$\mathbf{h}(t,\xi)\cdot\partial_{\mathbf{x}}$$

where $\mathbf{h}(t, \xi)$ are arbitrary solutions of system (5.8). The factorgroup is seven-parametric with respect to this normal divider. The basis of its Lie algebra consists of the operators

$$\partial_t, \quad \partial_\xi, \quad t\partial_t + 2\xi\partial_\xi, \quad \mathbf{x}\cdot\partial_{\mathbf{x}}, \quad \mathbf{x}\Lambda\partial_{\mathbf{x}}$$
$$2\xi\partial_t + t\mathbf{x}\Lambda\partial_{\mathbf{x}}, \quad 4\xi(t\partial_t + \xi\partial_\xi) + t^2\mathbf{x}\Lambda\partial_{\mathbf{x}} - 2\xi\mathbf{x}\cdot\partial_{\mathbf{x}}$$

The introduction of the complex dependent and independent variables

$$z = x^1 + ix^2, \alpha = i\xi, \beta = t \tag{5.9}$$

enables us to write system (5.8) in the form of a complex heat-conduction equation

$$z_\alpha = z_{\beta\beta} \tag{5.10}$$

A change to real variables using formulae (5.9) in the solution of Eq. (5.10), in which α and β are complex independent variables, while $\boldsymbol{z}=\boldsymbol{z}(\alpha, \beta)$ is a complex function, gives the solution of system (5.8). Consequently, Eq. (5.10) can be effectively used to obtain accurate solutions of system (5.8).

When $n=3$ system (5.4) with the four independent variables t, ξ^1, ξ^2, ξ^3 is reduced to the equivalent system of the form (5.7) with three independent variables $\boldsymbol{t}$, ξ^1, ξ^2

$$x^1_{tt} = -\left[x^2, x^3\right], \quad x^2_{tt} = -\left[x^3, x^1\right], \quad x^3_{tt} = -\left[x^1, x^2\right]; \quad \left(x^1, x^2, x^3\right) = \mathbf{x} \tag{5.11}$$

where $[a, b]=|\partial(a, b)|/\partial(\xi^1, \xi^2)|$ is the Poisson bracket.

The fundamental Lie group of transformations of system (5.11) is infinite and is generated by the operators

$$t\partial_t + \xi^1\partial_{\xi^1} + \xi^2\partial_{\xi^2}, \quad t\partial_t - 2\mathbf{x}\cdot\partial_{\mathbf{x}}, \quad \mathbf{x}\Lambda\partial_{\mathbf{x}}, \quad \partial_{\mathbf{x}}, \quad t\partial_{\mathbf{x}}, \quad g_{\xi^2}\partial_{\xi^1} - g_{\xi^1}\partial_{\xi^2}$$

where $g=g(\xi^1, \xi^2)$ is an arbitrary function.

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